MathWit Club Public tutorial Sample Questions 1

Exercise 0.1. *Decomposing a matrix, answer is given. Can you fill in the steps?*

$$\begin{pmatrix} 18 & -51 & 27 & -15 \\ 8 & -24 & 14 & -8 \\ 15 & -48 & 28 & -15 \\ 15 & -47 & 25 & -12 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 0 & 3 & 3 \\ 3 & 1 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & -9 & 5 & -3 \\ -5 & 15 & -9 & 6 \\ -1 & 2 & -1 & 1 \\ -1 & 4 & -2 & 1 \end{pmatrix}$$

When solving an equation, sometimes we need to eliminate root(s) that are created by some operations. The following is an example that contains an error.

Example 0.2. *Solve a simple equation* $x + \sqrt{x} = 2$.

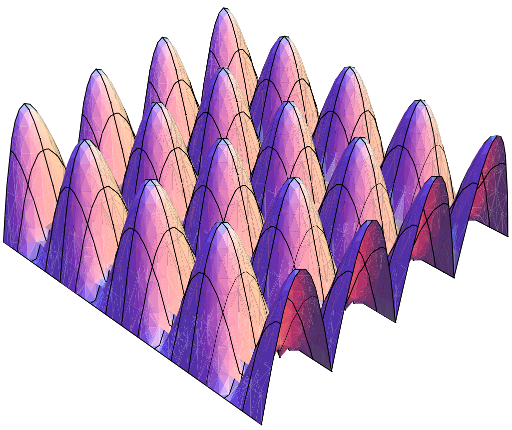
$$x + \sqrt{x} = 2 \Leftrightarrow \sqrt{x} = 2 - x \Leftrightarrow x = (2-x)^2 \Leftrightarrow x^2 - 5x + 4 = 0$$

Solve the quadratic equation on the right, we have $x = 1$ or $x = 4$.

As you can see, $x = 4$ is not a root for the original equation. The mistake is the second $\Leftrightarrow$ (if and only if). From left, it can derive the right, but not the other way around. If $2 - x$ is negative, the equation does not make sense, since square root, without a sign, is supposed to be positive (a convention). Unfortunately, $x = 4$ will make $2 - x$ negative, thus we need to eliminate this extra value, so $x = 1$ is the only solution.

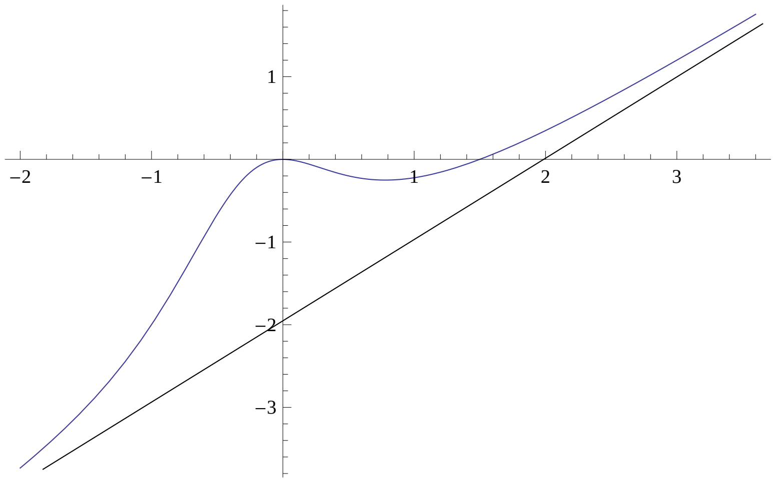
Exercise 0.3. *Discuss the range of the function and find the number of local maximums.*

$$f(x, y) = |\sin 2x \cos 2y|$$



Example 0.4. *Find the oblique asymptote of*

$$f(x) = \frac{4x^3 - 6x^2}{4x^2 + 2x + 3}.$$



Solution 1: Suppose the asymptote is $y = mx + b$,then (we assume $+\infty$, for limit to $-\infty$ we will have the same result).

$$\lim_{x\to\infty}\left(\frac{4x^3-6x^2}{4x^2+2x+3}-mx-b\right)=\lim_{x\to\infty}\frac{(4-4m)x^3-(6+2m+4b)x^2-(3m+2b)x-3b}{4x^2+2x+3}$$

The above limit should be zero, hence $4-4m=0 \Rightarrow m=1$ and $6+2m+4b=0 \Rightarrow b=-2$. So the asymptote is

$$y = x - 2.$$

Solution 2: First find the slope of asymptote:

$$\lim_{x\to\infty}\frac{y}{x}=\lim_{x\to\infty}\frac{4x^2-6x}{4x^2+2x+3}=1. \tag{1}$$

The above is justified because

$$\lim_{x\to\infty}\left(\frac{4x^3-6x^2}{4x^2+2x+3}-mx-b\right)=0$$

Divide the following expression

$$\frac{4x^3-6x^2}{4x^2+2x+3}-mx-b$$

by x, then take limit $x\to\infty$ and then use the previous limit, we can see (1) is true.

So the asymptote is of the form $y = x + b$.

$$\lim_{x\to\infty}\left(\frac{4x^3-6x^2}{4x^2+2x+3}-x\right)=\lim_{x\to\infty}\frac{-8x^2-3x}{4x^2+2x+3}=-2 \quad\Rightarrow\quad b=-2.$$

Thus we get the same result as solution 1.♡